Low symmetry patterns on magnetic fluids

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The pattern formation on the free surface of a magnetic fluid subjected simultaneously to a vertical and a horizontal magnetic field is investigated theoretically. In this anisotropic system planforms less symmetric then squares and hexagons arise. The relative stability of parallel ridges and asymmetric patterns, periodic on a rectangular or a rhombic lattice, is studied using a perturbative energy minimization procedure. Moreover, the interplay between the anisotropy and the broken up-down symmetry of the system is quantitatively analyzed.

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I. INTRODUCTION

Spontaneous formation of patterns is a quite general phenomenon in many different fields of physics, for instance, in hydrodynamics, crystal growth, and nonlinear optics. Accordingly much efforts [1] have been devoted to understand the universal mechanisms leading to certain patterns. In systems that are homogeneous and *isotropic* in two dimensions, symmetry breaking instabilities result usually in regular hexagons, squares, or stripes. But also "superlattice" patterns [2–4] and quasiperiodic structures [5,6] can be found and have drawn considerable attention in the past years. Nevertheless, often asymmetries are more interesting then symmetries and thus pattern formation in *anisotropic* systems came into the focus of nonlinear science. For instance, the convection in nematic liquid crystals with magnetic field was investigated [7] and in inclined layer convection [8] novel states were found. Recently, in an anisotropic optical system "squeezed" hexagons were observed [9].

The existence of the corresponding nonequilateral patterns was theoretically discussed in the framework of amplitude equations with broken rotational symmetry [10–12]. However, these previous investigations focused on the qualitative aspects and did not consider a concrete physical system for which the coefficients of the amplitude equations were calculated.

In contrast, in this paper we report a theoretical investigation of "stretched" hexagons and stretched squares in a definite physical system. It will be shown that in a tilted magnetic field nonequilateral patterns develop on a magnetic fluid. A magnetic fluid is a suspension of ferromagnetic nanoparticles in a suitable carrier liquid, which yields a superparamagnetic Newtonian fluid [13]. When the free surface of this ferrofluid is subjected to a vertical magnetic field, the normal field or Rosensweig instability [14] occurs above a certain threshold for the field and results in an array of fluid peaks [14,15]. By tilting the magnetic field, the left-right symmetry of this system can be broken in a controllable way. This should facilitate a comparison of our predictions with corresponding experiments.

Our theoretical investigation is a generalization of the energy variational method used in Ref. [15]. Near the instability threshold the energy functional of the system may be written as a power series in the amplitude of the surface deflection. By minimizing this expansion of the energy we can *quantitatively* investigate the formation of the stretched patterns.

Moreover, due to the left-right asymmetry the arising nonequilateral but periodic patterns lack any point symmetry and are in this sense the complements of quasiperiodic structures that are spatially aperiodic but have a definite rotational symmetry. We present the detailed shape and the stability of the stretched patterns.

In Sec. II the system is introduced and the basic equations are established. After transforming the equations into a form suitable for the energy calculation the outcomes of the perturbative variational procedure are shown in Sec. III. Section IV focuses on the effects of the anisotropy on the bifurcation structure. The concluding Sec. V contains a summary and a discussion of our results.

II. BASIC EQUATIONS

We consider a magnetic fluid subjected to a tilted magnetic field $\mathbf{H}_0$, which in the absence of any magnetic permeable material is homogeneous and of the form $\mathbf{H}_0 = H_Z \mathbf{e}_z + H_X \mathbf{e}_x$. The gravitational acceleration $\mathbf{g} = -g\mathbf{e}_z$ acts parallel to the z axis. The incompressible magnetic fluid with infinite depth and density ρ, permeability μ, and surface tension σ has a free surface described by $z = \zeta(x,y)$ with the magnetically impermeable air above. Our aim is to determine which static profile $\zeta(x,y)$ develops for a magnetic field strong enough to destabilize the flat surface $\zeta(x,y) = 0$ of the fluid.

Stable configurations of the surface with infinite horizontal extension are given by minima of the thermodynamic potential per unit area in the x-y plane. This potential comprises the hydrostatic energy, the magnetic energy [16], and the surface energy, respectively:

$$f[\zeta(x,y)] = \Big\langle \frac{\rho g}{2}\zeta^2(x,y) - \frac{\mu-\mu_0}{2}\int_{-\infty}^{\zeta(x,y)} dz\, \mathbf{H}_0 \mathbf{H}(x,y,z) + \sigma\sqrt{1+[\partial_x \zeta(x,y)]^2 + [\partial_y \zeta(x,y)]^2} \Big\rangle. \qquad (1)$$

Here the brackets denote the average over the x-y plane, μ_0

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is the permeability of free space, and $\mathbf{H}(x,y,z)$ is the magnetic field in the presence of the magnetic fluid.

The magnetic fields $\mathbf{B}$ and $\mathbf{H}$ are determined by the static Maxwell equations

$$\nabla\cdot\mathbf{B}=0 \quad \text{and} \quad \nabla\times\mathbf{H}=\mathbf{0}, \tag{2}$$

together with the appropriate boundary conditions at the fluid surface $z=\zeta(x,y)$ and the condition

$$\lim_{z\to+\infty}\mathbf{H}(x,y,z)=H_Z\mathbf{e}_z+H_X\mathbf{e}_x. \tag{3}$$

Throughout the paper we will assume the linear relation

$$\mathbf{B}=\mu_0\mu_\mathrm{r}\mathbf{H} \tag{4}$$

between the magnetic induction and the magnetic field, where $\mu_r=\mu/\mu_0$ denotes the relative permeability of the magnetic fluid.

The two characteristic scales of the problem are the critical magnetic field at the onset of the instability,

$$H_\mathrm{c}=\sqrt{\frac{2\mu_\mathrm{r}(\mu_\mathrm{r}+1)\sqrt{\rho g\sigma}}{\mu_0(\mu_\mathrm{r}-1)^2}}, \tag{5}$$

and the corresponding critical wave number $k_\mathrm{c}=\sqrt{\rho g/\sigma}$, which were first derived in Ref. [14] from a linear stability analysis. Henceforth we measure all lengths in units of the capillary length k_c^{-1}, all wave numbers in units of the critical wave number k_c, and energies per unit area in units of the surface tension σ.

Moreover, it is convenient to introduce a dimensionless scalar magnetic potential $\psi(x,y,z)$, defined by $\mathbf{H}=(H_Z/k_\mathrm{c})\nabla\psi$, which has to satisfy the Laplace equation

$$\Delta\psi=0 \tag{6}$$

in the magnetic fluid and in the air above. At $z=\zeta(x,y)$ the continuity of the component of $\mathbf{B}$ normal to the surface and of the component of $\mathbf{H}$ tangential to the surface requires

$$\begin{aligned}\mu_\mathrm{r}\lim_{\epsilon\to+0}&[(\partial_x\zeta)\partial_x\psi+(\partial_y\zeta)\partial_y\psi-\partial_z\psi]|_{z=\zeta-\epsilon}\\ &=\lim_{\epsilon\to+0}[(\partial_x\zeta)\partial_x\psi+(\partial_y\zeta)\partial_y\psi-\partial_z\psi]|_{z=\zeta+\epsilon}\end{aligned} \tag{7}$$

and

$$\lim_{\epsilon\to0}\psi|_{z=\zeta-\epsilon}=\lim_{\epsilon\to0}\psi|_{z=\zeta+\epsilon}, \tag{8}$$

respectively. The asymptotic form of the magnetic field for $z\to+\infty$ as specified by Eq. (3) translates into the condition

$$\lim_{z\to+\infty}\nabla\psi=\mathbf{e}_z+\frac{H_X}{H_Z}\mathbf{e}_x. \tag{9}$$

Exploiting the fact that $\mathbf{H}_0$ is homogeneous we finally get the energy f as a functional of the surface deflection $\zeta=\zeta(x,y)$ in the form

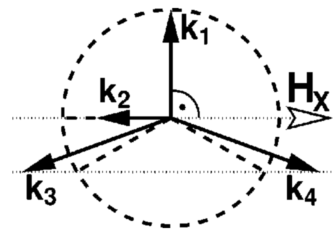

FIG. 1. Wave vectors $\mathbf{k}_1$ to $\mathbf{k}_4$ of the four main modes in the perturbation ansatz (11). $\mathbf{k}_1$ is perpendicular to $\mathbf{k}_2$ and to the horizontal magnetic field $H_X\mathbf{e}_x$. $\mathbf{k}_3$ and $\mathbf{k}_4$ have the same modulus and satisfy the resonance condition $\mathbf{k}_1+\mathbf{k}_3+\mathbf{k}_4=\mathbf{0}$.

$$\begin{aligned}f[\zeta]=\Bigg\langle&\frac{\zeta^2}{2}-\psi\Big|_{z=\zeta}\frac{\mu_\mathrm{r}(\mu_\mathrm{r}+1)H_Z^2}{(\mu_\mathrm{r}-1)H_\mathrm{c}^2}\left(1-\frac{H_X}{H_Z}\partial_x\zeta\right)\\ &+\sqrt{1+(\partial_x\zeta)^2+(\partial_y\zeta)^2}\Bigg\rangle.\end{aligned} \tag{10}$$

An exact minimization of the energy functional $f[\zeta]$ is, in general, impossible due to the rather implicit dependence of the potential $\psi(x,y,z)$ on the surface deflection $\zeta(x,y)$ specified by the boundary conditions (7) and (8). In order to make analytic progress, we restrict ourselves to the vicinity of the critical magnetic field and assume that the amplitude of the surface deflection is still small. It is then possible to expand the energy in this amplitude and to retain only the first terms. In this paper we will consider an expansion of the energy up to fourth order in the amplitude.

Generalizing the perturbation expansions used in Ref. [15] we write the surface profile in the form

$$\zeta(x,y)=\frac{1}{2}\sum_{n=1}^{17}A_ne^{i(\mathbf{k}_n\cdot\mathbf{r})}+\text{c.c.}, \tag{11}$$

where $\mathbf{r}=(x,y)$ and $\mathbf{k}=(k_x,k_y)$ are two-dimensional vectors. The wave vectors $\mathbf{k}_1=(0,k),\mathbf{k}_2=(-q,0),\mathbf{k}_3=(-p\sqrt{3}k/2,-k/2)$, and $\mathbf{k}_4=(p\sqrt{3}k/2,-k/2)$ of the main modes $n=1,\ldots,4$ are shown in Fig. 1. To allow the wave numbers of the main modes to deviate from each other we have introduced the additional free parameters k, q, and p. This extension is essential since the magnetic field H_X explicitly breaks the rotational invariance of the energy f, as can be seen easily from Eq. (10). In the special case $k=q=p$ our ansatz allows the description of regular squares and hexagons as they arise from the normal field instability (i.e., $H_X=0$). In the presence of an additional and sufficiently strong horizontal magnetic field, also fluid ridges (stripes) with axes parallel to this field appear on the surface [17]. Accordingly the wave vector $\mathbf{k}_1$ is oriented perpendicular to the horizontal field $H_X\mathbf{e}_x$.

In our ansatz (11) the terms with $n=5,\ldots,17$ are higher harmonics with wave vectors $\mathbf{k}_5$–$\mathbf{k}_{17}$ being linear combinations of two wave vectors of the main modes. The amplitudes of these additional modes are of order $O(A_n^2)$ in the main modes $n=1,\ldots,4$. The higher harmonics must be included in the investigation of the weakly nonlinear regime since they contain information on the deviation from the simple cosine shape describing the linear instability. Under the force of a tilted magnetic field (i.e., $H_X \neq 0$) such a deviation may result in the formation of patterns of asymmetric peaks. This asymmetry of the shape was found earlier in corresponding experiments with a single ferrofluid drop [18] and is a consequence of the broken left-right symmetry, $x \to -x$, of the energy functional f with a horizontal field. For this reason at least the phases ϕ_n with $n=3,\ldots,17$ of the complex valued and unknown amplitudes A_n must be included into the set of variational parameters. Due to the translational invariance of the energy functional f the phases ϕ_1 and ϕ_2 can be assumed to be zero.

By means of a suitable ansatz for the magnetic potential with a similar dependence on x and y as $\zeta(x,y)$ it is possible to determine $\psi(x,y,z)$ perturbatively as a function of the free parameters $\{k,q,p,\phi_n,|A_n|\}$. Using Eq. (10) we then calculate the energy as a function of these variational parameters up to the fourth order in the main amplitudes. The minimization in the higher-order amplitudes A_n with $n>4$ can be performed explicitly, and after introducing the supercriticality parameter

$$\epsilon = \frac{H_Z^2}{H_c^2} - 1, \tag{12}$$

we finally arrive at

$$\begin{aligned} f = & -\tfrac{1}{2}\{l_1(\epsilon)|A_1|^2 + l_2(\epsilon)|A_2|^2 + l_3(\epsilon)[|A_3|^2+|A_4|^2]\} \\ & -\tfrac{1}{2}\gamma[A_1A_3A_4 + A_1^*A_3^*A_4^*] + \tfrac{1}{4}(g_1|A_1|^4 + g_2|A_2|^4 \\ & + g_3[|A_3|^4+|A_4|^4]) + \tfrac{1}{2}\, g_{1,2}|A_1|^2|A_2|^2 \\ & + \tfrac{1}{2}\, g_{1,3}|A_1|^2[|A_3|^2+|A_4|^2] + \tfrac{1}{2}\, g_{3,4}|A_3|^2|A_4|^2 \\ & + \tfrac{1}{2}\, g_{2,3}|A_2|^2[|A_3|^2+|A_4|^2]. \end{aligned} \tag{13}$$

The coefficients of this expansion are functions of μ_r, H_X^2, and the variational parameters k,q, and p. The analytic expressions are too long to be stated here and are therefore given in Ref. [19]. Due to the complicated dependence of the coefficients on the wave numbers, our result for the energy $f(k,q,p,A_{1,\ldots,4})$ has to be minimized numerically.

III. RESULTS

Exemplarily some resulting surface profiles are displayed in Fig. 2. In the case of the normal field instability, either a square or a hexagonal pattern of fluid peaks arise for overcritical magnetic fields. If the magnetic fluid is, in contrast, subjected simultaneously to an overcritical vertical and a horizontal field, our analysis reveals that stable surface profiles are less symmetric. Then the energy f attains a minimum at $|A_1|>A_2=A_3=A_4=0$ (parallel ridges) or at $|A_1|>|A_2|>A_3=A_4=0$ (two-mode solution) and at $|A_1|>|A_3|=|A_4|>A_2=0$ (three-mode solution). For these solutions the amplitude A_1 is always larger then the amplitudes of the other main modes since the horizontal field depresses the modes $n=2,\ldots,4$. Thus the corresponding patterns have neither fourfold nor threefold rotational symmetry.

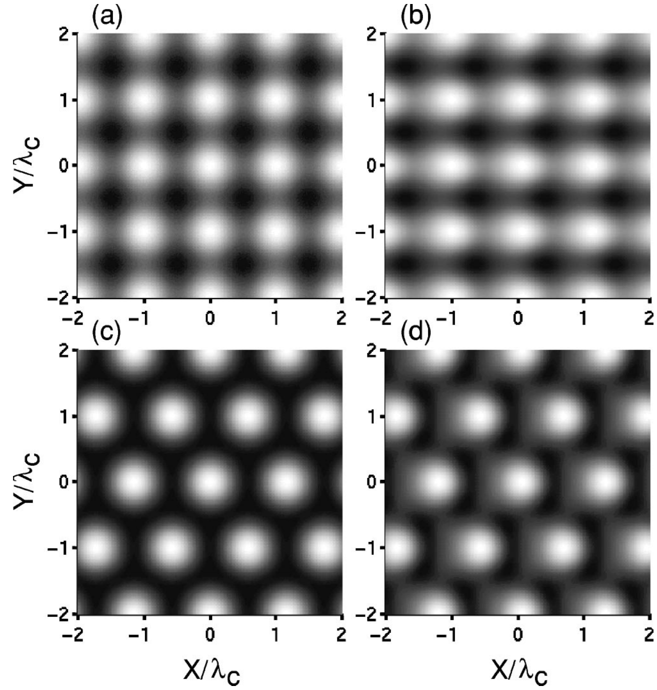

FIG. 2. Stable surface profiles of a magnetic fluid with relative permeability $\mu_r=1.3$ for different magnetic fields. High surface regions are brighter. For the vertical field $H_Z=1.017H_c$ the square pattern (a) is stable. The superposition $H_Z=1.017H_c$ and $H_X=0.15H_c$ leads to pattern (b), which is periodic on a rectangular lattice. The hexagonal pattern (c) arises for $H_Z=1.014H_c$. The profile (d) results from $H_Z=1.014H_c$ and $H_X=0.15H_c$ and is periodic on a rhombic lattice. $\lambda_c=2\pi/k_c$ is typically of the order of 10 mm.

Moreover, in Fig. 2 the solutions with two (b) and three (d) main modes are stretched in the direction of the horizontal magnetic field. This is a result of the minimization in q and p, which leads in all cases with $H_X\neq 0$ to values smaller then $k\approx 1$ for q and p. Accordingly the two-mode solution is merely periodic on a rectangular lattice and the three-mode solution on a rhombic lattice.

The peaks arising in the presence of a horizontal field (see Fig. 3) lack also the left-right symmetry, $x\to -x$. As expected the fluid peaks are tilted roughly parallel to the mag-

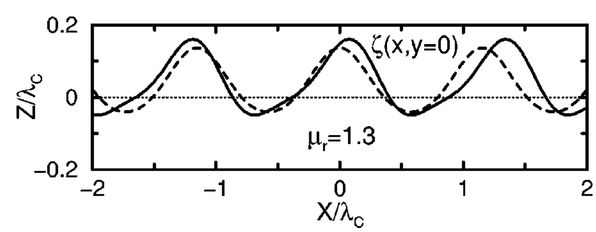

FIG. 3. Cross section $\zeta(x,y=0)$ through the surface profiles shown in Figs. 2(c) and 2(d). The normal field instability $H_Z=1.014H_c$ (dashed line) results in symmetric fluid peaks. In the tilted field $H_Z=1.014H_c$ and $H_X=0.15H_c$ (full line) the peaks are asymmetric and their mutual distance is larger. Moreover, the total amplitude of the surface deflection increases due to the enlarged absolute value H_0 of the magnetic field.

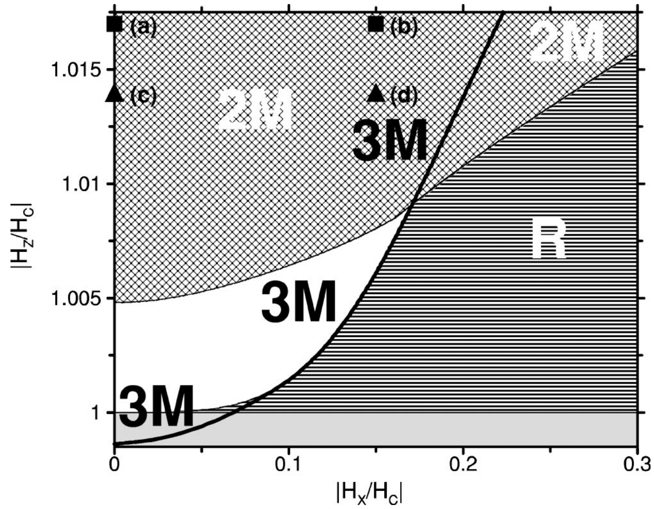

FIG. 4. Stability regions in the H_X-H_Z plane for a fluid with $\mu_r = 1.3$. Ridges (R) are stable in the region hatched with stripes and the two-mode solution ($2M$) is stable in the crosshatched region. In the area to the left of the thick line the three-mode solution ($3M$) is stable. Note that in certain sections of this area the system is bistable. The triangles and squares indicate the field combinations leading to the profiles shown in Fig. 2.

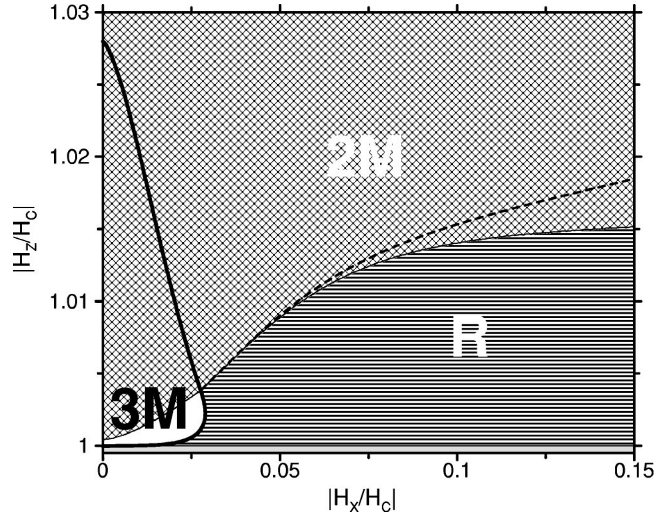

FIG. 5. The same as Fig. 4 but for a fluid with $\mu_r = 1.03$. Due to the smaller relative permeability, the parameter which rules the up-down symmetry breaking, the stability region of the three-mode solution shrinks. The dashed line indicates the limit between the stability regions of the two-mode and the ridge solution as they result, if a variation of the wave numbers k, q, and p is not taken into account.

netic field $\mathbf{H}_0$. This is related to the fact that the transformation $H_X/H_Z \rightarrow -H_X/H_Z$ requires the simultaneous reflection $\zeta(x,y) \rightarrow \zeta(-x,y)$ to leave the energy f unchanged, whereas f is invariant under the inversion $\mathbf{H}_0 \rightarrow -\mathbf{H}_0$. Hence the shape $\zeta(x,y)$ depends on the sign of H_X/H_Z but the stability of a solution is determined only by the absolute values of H_X and H_Z.

To predict whether an asymmetric pattern or parallel ridges will show up for a particular magnetic field $\mathbf{H}_0$ we address the pattern selection problem by studying the character of the extremum of the energy functional (10) at the two-mode, three-mode, and ridge solutions. The resulting stability regions in the vicinity of the critical field $H_Z \approx H_c$ for a fluid with relative permeability $\mu_r = 1.3$ are displayed in Fig. 4. For instance, in the normal magnetic field $H_Z = 1.014H_c$ at first a hexagonal pattern shows up (c). If, additionally, a horizontal magnetic field $H_X = 0.15H_c$ is applied, the regular hexagons are deformed and a pattern periodic on a rhombic lattice appears (d). An increase of the horizontal field up to $H_X = 0.21H_c$ destabilizes this three-mode solution and leads to a two-mode solution which is periodic on a rectangular lattice. A further increase of H_X eventually results in ridges parallel to the horizontal field. In contrast, due to an enlarged vertical field H_Z these ridges may become unstable and split up into peaks, which are again arranged on a rectangular or rhombic lattice. Thus ridges lose their stability in a sufficiently strong vertical magnetic field, whereas both two-dimensional structures can be destabilized by increasing the horizontal field induced anisotropy.

On the other hand in Fig. 5 particularly the stability region of the three-mode solution is reduced by decreasing the parameter μ_r, which rules the up-down symmetry breaking in the system. To show that the relative permeability μ_r determines whether the arising peaks (the center of the cells) point upward or downward we consider the situation without a horizontal field: Keeping in mind that $H_c^2(1/\mu_r) = H_c^2(\mu_r)/\mu_r$ it can be seen from Eq. (10) that the energy f is only invariant under the combined transformation $\zeta(x,y) \rightarrow -\zeta(x,y)$ *and* $\mu_r \rightarrow 1/\mu_r$. Note that with this transformation the magnetic potential $\psi(x,y,z)$ has to be replaced by $-\mu_r\psi(x,y,-z)$ since the condition (7) changes accordingly.

From the stability chart in Fig. 5 it can be seen that we reproduce the classical result for the stability of squares and hexagons [15], which was obtained only for a vertical field and fluids with small permeability $|\mu_r - 1| \ll 1$. But for the corresponding values of μ_r we found that a slight declination of the magnetic field $\mathbf{H}_0$ causes a remarkable reduction of the stability margins of the two-dimensional patterns, especially of the three-mode solution.

IV. BIFURCATIONS

In this section we consider the diverse bifurcation scenarios of the system to show that the transitions between the patterns are changed by the horizontal magnetic field. The general bifurcation structure of distorted or nonequilateral hexagons was studied in Ref. [12] using the center manifold reduction technique. To demonstrate the ''unfolding'' of the transcritical bifurcation we have determined for three different situations the surface deflection $\zeta(0,0)$ as a function of the supercriticality parameter ϵ. Since the surface profiles $\zeta(x,y)$ were calculated under consideration of the higher harmonics, the following bifurcation diagrams are not mirror symmetric for all solutions. This results from the broken up-down symmetry $\zeta(x,y) \rightarrow -\zeta(x,y)$ of the system.

The familiar bifurcation scenario for a fluid in a strictly vertical magnetic field is displayed in Fig. 6. In the diagram,

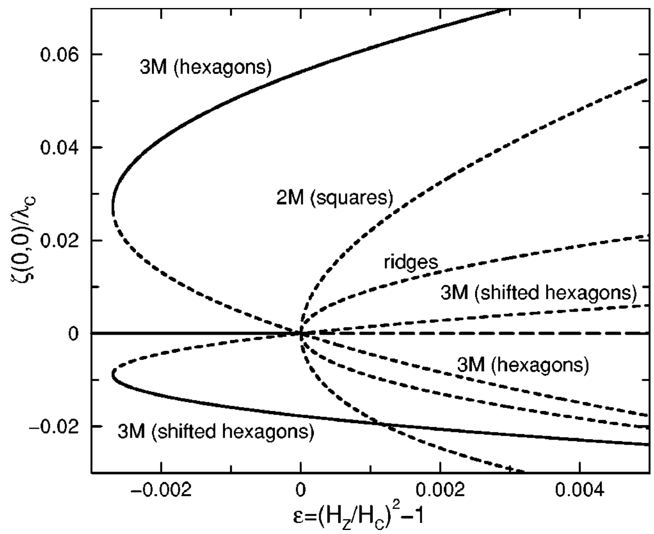

FIG. 6. Bifurcation diagram for a magnetic fluid with $\mu_r=1.3$ in the absence of a horizontal magnetic field $(H_X=0)$. The dependence of the surface deflection $\zeta(0{,}0)$ on the supercriticality parameter $\epsilon=H_Z^2/H_c^2-1$ is shown. Solid lines indicate stable surface profiles and dashed lines indicate unstable solutions. The stable hexagons have in their center fluid peaks pointing upward. The curve which belongs to the shifted hexagons describes the deflection $\zeta(0{,}0)$, if the three-mode solution is displaced by one wavelength λ_c/k in the y direction.

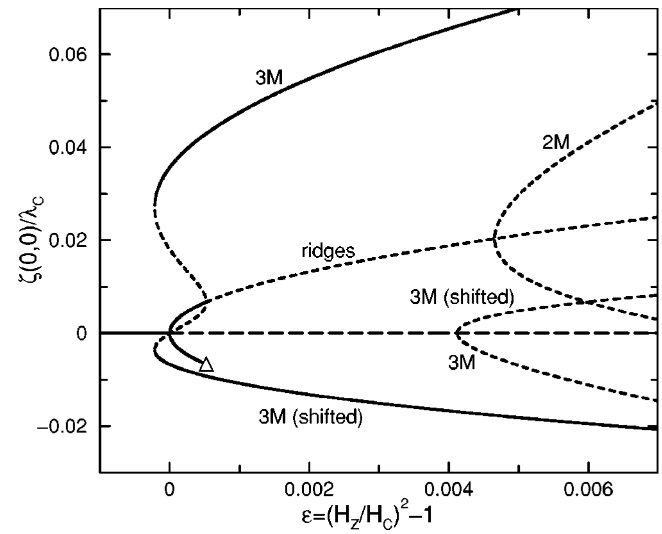

FIG. 7. Bifurcation diagram for the same fluid as in Fig. 6 but in the presence of a tilted magnetic field. Due to the horizontal field of the constant value $H_X=0.065H_c$, parallel ridges are stable for weakly supercritical fields H_Z. The lower branch of the ridge solutions splits up at the point marked with the triangle. Since this branch describes the depth of the valleys between the ridges or simply the patterns displaced by $\lambda_c/(2k)$ in the y direction its further bifurcations can be deduced from the upper part of the diagram.

the range of the supercriticality parameter ϵ corresponds to $0.9985<H_Z/H_c<1.0025$ in Fig. 4 $(H_X/H_c=0)$. At $\epsilon=0$ the trivial solution splits up by two supercritical pitchfork bifurcations into unstable two-mode solutions and unstable ridges. In contrast, the unstable three-mode solutions appear at a transcritical bifurcation. But only hexagons with fluid peaks pointing upward are stable. The corresponding stable branch disappears together with the unstable branch of the three-mode solutions for subcritical fields $H_Z<H_c$ at a saddle-node bifurcation. The dependence of the surface deflection $\zeta(0{,}0)$ of the hexagonal patterns shifted by λ_c/k in the y-direction can be calculated by changing simultaneously the sign of the amplitudes A_3 and A_4. These three-mode solutions are topological identical to the unshifted hexagons.

In Fig. 7 the bifurcation scenario is clearly changed by the additional horizontal field $H_X=0.065H_c$. At $\epsilon=0$ only ridges appear from the trivial solution at a supercritical pitchfork bifurcation. The corresponding solutions are initially stable but lose stability at a subcritical pitchfork bifurcation to the three-mode solutions. For larger values of H_Z the unstable branch of the ridges splits up at a supercritical pitchfork bifurcation into two unstable two-mode solutions, which are both identical except for a translation by $\lambda_c/(2q)$ in the x direction. Since the three-mode solutions bifurcate subcritically, the ridges and the three-mode solutions are simultaneously stable in a certain region (cf. Fig. 4).

For magnetic fluids with small permeability the impact of the horizontal magnetic field becomes even more pronounced. In Fig. 8 two separate intervals for the vertical field H_Z exist, in which ridges are stable. For the fluid with $\mu_r=1.03$ the ridges lose stability to the three-mode solutions at a supercritical pitchfork bifurcation. A corresponding change from a subcritical to a supercritical bifurcation due to a decrease of the permeabilty μ_r was also found in Refs. [20,21]. The three-mode solutions lose stability at a saddle-node bifurcation and rejoin the ridge branch, at which point the ridges regain stability. Eventually stable two-mode solutions bifurcate from the ridge branch. For horizontal fields $H_X<0.0273H_c$ this supercritical pitchfork bifurcation takes

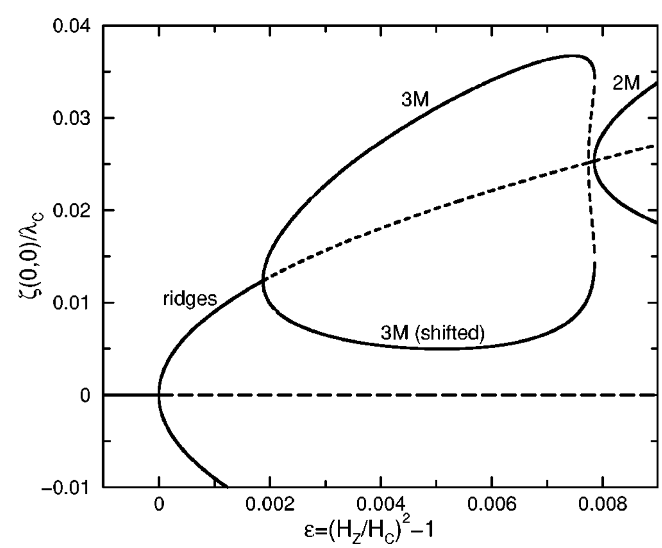

FIG. 8. Bifurcation diagram for a magnetic fluid with small permeability $\mu_r=1.03$ in the presence of a tilted field. In the considered case with the horizontal magnetic field $H_X=0.0273H_c$, parallel ridges are stable both for weakly supercritical fields and for fields $H_Z\approx1.0039H_c$ (cf. Fig. 5). Due to the small relative permeability the three-mode solutions bifurcate supercritically from the ridges.

place before the three-mode solutions rejoin the ridge branch so that a direct transition from a three-mode solution to a two-mode solution can be observed (cf. Fig. 5).

V. DISCUSSION

In this paper we have theoretically investigated the pattern formation on a magnetic fluid subjected to a homogeneous magnetic field with arbitrary orientation. In contrast to previous theoretical papers [10–12], in which the existence of nonequilateral patterns was deduced from model amplitude equations, we have studied the formation of stretched patterns by means of an energy minimization principle. Our quantitative results can be compared directly with corresponding experiments.

We have shown that the magnetic field induced surface instability can result in parallel ridges and patterns periodic on a hexagonal, rhombic, square, or rectangular lattice. This rich spectrum of patterns and the fact that it can be explored by varying just the parameter of anisotropy, i.e., the horizontal magnetic field, is to our knowledge unique. Solely the transition from hexagons via rhomboids to stripes could be demonstrated so far in an anisotropic optical system [9].

In addition, we have found that the patterns that are stretched in the direction of the horizontal magnetic field lack any (nontrivial) rotational symmetry. For that reason these deformed but strictly periodic structures can be viewed as the counterparts of quasiperiodic patterns [5,6]. Unfortunately our weakly nonlinear analysis is restricted to the investigation of slightly deformed patterns in a moderately tilted field. Though it is well known that a horizontal magnetic field retards deformations of the flat surface of a magnetic fluid [13,22], the peaks resulting from the normal field instability become larger if they are subjected additionally to a horizontal field (see Fig. 3). This nonlinear effect basically sets the limits of the presented perturbative investigation. Nevertheless our study indicates that very oblique magnetic fields may result in highly stretched patterns.

Moreover, our analysis reveals the role of the up-down symmetry breaking in the anisotropic system. We have found a strong dependence of the pattern selection on the relative permeability μ_r of the magnetic fluid (cf. Figs. 4 and 5). In particular, for $|\mu_\mathrm{r}-1|\ll 1$ the stability of the patterns is rather susceptible to perturbations of the isotropy. We therefore conclude that even a slight disturbance of the rotational symmetry of a pattern forming system may lead to unexpected planforms, especially if the up-down symmetry of the system is only weakly broken. This is quite relevant, since the up-down symmetry breaking coefficient in the amplitude equations describing a pattern selection problem is usually assumed to be small.

The diverse bifurcation scenarios additionally demonstrate how the transitions between patterns are influenced by the horizontal magnetic field. The corresponding effects of anisotropy on the general bifurcation structure of distorted hexagonal patterns were found earlier in Ref. [12]. In a complementary study of a specific anisotropic system, the bifurcations of patterns during directional solidification were addressed [23]. However, the pattern forming system investigated in the present paper has the considerable advantage that a very precise measuring and tuning of the parameter that rules the up-down asymmetry, i.e., the relative permeability, and the parameter that rules the left-right asymmetry, i.e., the horizontal magnetic field, should be feasible.

Finally, we note that in Fig. 3 the asymmetric peaks look similar to the drifting peaks discussed in Ref. [18]. Thus we close with the interesting question of whether the interplay of dissipation, the up-down asymmetry, and the left-right asymmetry caused by the tilted field can generate two-dimensional drifting patterns. For compressible magnetoconvection in an oblique magnetic field it was shown that the combination of these asymmetries gives rise to traveling waves [24,25]. A possible modification of the static setup considered in the present paper to a dynamic system might be a parametrically driven magnetic fluid in a vibrated vessel [26] or a modulated magnetic field [4]. To describe these nonequilibrium systems our variational approach has to be extended taking into account the influence of inertia and damping [18,27].

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